

Symmetries, psudosymmetries and conservation laws in Lagrangian and Hamiltonian k -symplectic formalisms

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Abstract

In this paper we will present Lagrangian and Hamiltonian k -symplectic formalisms, we will recall the notions of symmetry and conservation law and we will define the notion of pseudosymmetry as a natural extension of symmetry. Using symmetries and pseudosymmetries, without the help of a Noether type theorem, we will obtain new kinds of conservation laws for k -symplectic Hamiltonian systems and k -symplectic Lagrangian systems.

AMS Subject Classification (2010): 70S05, 70S10, 53D05.

Key words: symmetry, pseudosymmetry, conservation law, Noether theorem, k -symplectic Hamiltonian system, k -symplectic Lagrangian system.

1 Introduction

The k -symplectic formalism ([3], [4], [10], [19]) is the generalization to field theories of the standard symplectic formalism in autonomous Mechanics, which is the geometric framework for describing autonomous dynamical systems ([1], [2]). This formalism is based on the polysymplectic formalism developed by Günther ([10]). The k -Symplectic Geometry provides the simplest geometric framework for describing certain class of first-order classical field theories. Using this description we analyze different kinds of symmetries for the Hamiltonian and Lagrangian formalisms of these field theories, including the study of conservation laws associated to them and stating Noether's Theorem ([27], [5], [28]). Further more, we will generalize the study of symmetries and conservation laws from classical (symplectic, $k = 1$) formalism to the k -symplectic formalism for obtain new kinds of conservation laws for k -symplectic Hamiltonian and Lagrangian systems. A similar study for the case of higher order tangent bundles geometry was done by the author in [18], [20].

In this paper we will revisited the study of symmetries, conservation laws and relationship between this in the framework of k -symplectic geometry and we

will improved the results obtained in [21] and [22]. More exactly, we intend to extend the study of symmetries and conservation laws from Classical Mechanics to the first-order classical field theories, both for the Lagrangian and Hamiltonian formalisms, using Günther's k -symplectic description, and considering only the regular case. We will find new kinds of conservation laws, nonclassical, without the help of a Noether's type theorem, using only the relationship between symmetries, pseudosymmetries and conservation laws.

The study of symmetries and conservation laws for k -symplectic Hamiltonian systems is, like in the classical case, a topic of great interest and was developped recently by M. Salgado, N. Roman-Roy, S. Vilarino in [27], [28] and L. Bua, I. Bucătaru, M. Salgado in [5]. Further more, in the paper [16] J.C. Marrero, N. Roman-Roy, M. Salgado, S. Vilarino begin the study of symmetries and conservation laws for k -cosymplectic Hamiltonian systems, like an extension to field theories of the standard cosymplectic formalism for nonautonomous mechanics ([13], [14]). In [27] the Noether's theorem, obtained for a k -symplectic Hamiltonian system, associates conservation laws to so-called Cartan symmetries. However, these kinds of symmetries do not exhaust the set of symmetries. As is known, in mechanics and physics there are symmetries which are not of Cartan type, and which generate conserved quantities, i.e. conservation laws ([15], [25], [26]).

In the second section are presented the basic tools and, also the classical result who will be generalized in the last section. Following [27], [28], in sections three and four we will review the essential geometric elements of k -symplectic formalism who need us to explain and to obtain the results from the last section. So, by generalization from symplectic geometry to k -symplectic geometry, we will obtain new kinds of conservation laws for k -symplectic Hamiltonian and Lagrangian systems, without the help of a Noether type theorem and without the use of a variational principle. The main result is a generalization from the classical case ($k = 1$) of a results of G.L. Jones ([11]) and M. Crăsmăreanu ([7]).

All manifolds and maps are C^∞ . Sum over crossed repeated indices is understood.

2 Basic tools

Let M be a n -dimensional manifold, $C^\infty(M)$ the ring of real-valued functions, $\mathcal{X}(M)$ the Lie algebra of vector fields and $A^p(M)$ the $C^\infty(M)$ -module of p -forms, $1 \leq p \leq n$.

Let us recall that if Δ is a distribution with a constant rank k on M and $\Phi : M \rightarrow M$ is a diffeomorfism of M , then Φ is called an *invariant transformation* or *finite symmetry* of Δ if for all $x \in M$, $T\Phi(\Delta_x) \subset \Delta_{\Phi(x)}$ ([12]).

If $\{\Phi_t^\xi\}_t$ denote the local one-parameter group of transformations of the vector field ξ on M , then ξ is called *symmetry* or *infinitesimal symmetry* or *dynamical symmetry* of Δ if for all t , $\{\Phi_t^\xi\}_t$ is an invariant transformation of Δ .

ξ is a symmetry of Δ if and only if for all $\zeta \in \Delta$, $[\xi, \zeta] \in \Delta$, or equivalently, the local flow of ξ transfer integral mappings in integral mappings and conse-

quently, for any integral manifold Q of Δ , $\{\Phi_t^\xi\}_t(Q)$ is another integral manifold of Δ .

A function $g : U \rightarrow \mathbf{R}$ (U being an open subset of M) is called *first integral* or *conservation law* of Δ if the one-form dg belongs to Δ , i.e. $i_\xi dg = 0$, for all $\xi \in \Delta$.

If g is a first integral of Δ on U and Q is an integral manifold of Δ with integral mapping $i : Q \rightarrow U \subseteq M$, then $d(g \circ i) = 0$, that means the function g is constant along the integral manifold Q .

For $X \in \mathcal{X}(M)$ with local expression $X = X^i(x) \frac{\partial}{\partial x^i}$ we consider the system of ordinary differential equations which give the flow $\{\Phi_t\}_t$ of X , locally,

$$\dot{x}^i(t) = \frac{dx^i}{dt}(t) = X^i(x^1(t), \dots, x^n(t)). \quad (2.1)$$

A *dynamical system* is a couple (M, X) . A dynamical system is denoted by the flow of X , $\{\Phi_t\}_t$ or by the system of differential equations (2.1).

A function $f \in C^\infty(M)$ is called *conservation law* for dynamical system (M, X) if f is constant along the every integral curves of X (solutions of (2.1)), that is

$$L_X f = 0, \quad (2.2)$$

where $L_X f$ means the Lie derivative of f with respect to X .

If $Z \in \mathcal{X}(M)$ is fixed, then $Y \in \mathcal{X}(M)$ is called *Z-pseudosymmetry* for (M, X) if there exists $f \in C^\infty(M)$ such that $L_X Y = fZ$. A *X-pseudosymmetry* for X is called *pseudosymmetry* for (M, X) . $Y \in \mathcal{X}(M)$ is called *symmetry* for (M, X) if $L_X Y = 0$. Recall that $\omega \in A^p(M)$ is called *invariant form* for (M, X) if $L_X \omega = 0$.

Example 2.1. ([8], [9]) For Nahm's system from the theory of static SU(2)-monopoles:

$$\frac{dx^1}{dt} = x^2 x^3, \quad \frac{dx^2}{dt} = x^3 x^1, \quad \frac{dx^3}{dt} = x^1 x^2, \quad (2.3)$$

the vector field $Y = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3}$ is a pseudosymmetry.

The notion of pseudosymmetry defined above is a weaker notion of symmetry. This is a natural generalization of the notion of symmetry for a system of ordinary differential equations (2.1). Symmetries and pseudosymmetries are just infinitesimal symmetries of the distribution generated by the vector field X ([12]).

The next theorem which gives the association between pseudosymmetries and conservation laws is due to M. Crăsmăreanu ([7]) and G.L. Jones ([11]). We will find new kinds of conservation laws, nonclassical, without the help of Noether's type theorem.

Theorem 2.2. *Let $X \in \mathcal{X}(M)$ be a fixed vector field and $\omega \in A^p(M)$ be a invariant p -form for X . If $Y \in \mathcal{X}(M)$ is symmetry for X and $S_1, \dots, S_{p-1} \in \mathcal{X}(M)$ are $(p-1)$ Y -pseudosymmetry for X then*

$$\Phi = \omega(X, S_1, \dots, S_{p-1}) \quad (2.4)$$

or, locally,

$$\Phi = S_1^{i_1} \cdots S_{p-1}^{i_{p-1}} Y^{i_p} \omega_{i_1 \dots i_{p-1} i_p} \quad (2.5)$$

is a conservation laws for (M, X) .

Particularly, if Y, S_1, \dots, S_{p-1} are symmetries for X then Φ given by (2.4) is conservation laws for (M, X) .

If (M, ω) is a symplectic manifold then the dynamical system (M, X) is said to be a *dynamical Hamiltonian system* if there exists a function $H \in C^\infty(M)$ (called the *Hamiltonian*) such that $i_X \omega = -dH$, where i_X denotes the interior product with respect to X .

It is known that the symplectic form ω is an invariant 2-form for (M, X) and the Hamiltonian H is a conservation law for (M, X) .

Now, we can apply Theorem 2.2 to the dynamical Hamiltonian systems.

Proposition 2.3. *Let be (M, X_H) a Hamiltonian system on the symplectic manifold (M, ω) , with the local coordinates (x^i, p_i) . If $Y \in \mathcal{X}(M)$ is a symmetry for X_H and $Z \in \mathcal{X}(M)$ is a Y -pseudosymmetry for X_H then*

$$\Phi = \omega(Y, Z) \quad (2.6)$$

is a conservation law for the Hamiltonian system (M, X_H) .

Particularly, if Y and Z are symmetries for X_H then Φ from (2.6) is a conservation law for (M, X_H) .

3 k -Symplectic Hamiltonian formalism

Let $(T_k^1)^*M = T^*M \oplus \dots \oplus T^*M$ (the Whitney sum of k copies of T^*M) be the k -cotangent bundle of a n -dimensional differentiable manifold M , with the projection $\tau^* : (T_k^1)^*M \rightarrow M$. The natural coordinates on $(T_k^1)^*M$ are (x^i, p_i^A) , $1 \leq i \leq n, 1 \leq A \leq k$.

The *canonical k -symplectic structure* in $(T_k^1)^*M$ is (ω_A, V) , where $V = \ker(\tau^*)^*$ and $\omega_A = (\tau_A^*)^* \omega = -d(\tau_A^*)^* \theta = -d\theta_A$. $\omega = -d\theta$ is the canonical symplectic structure in T^*M , θ is the Liouville 1-form in T^*M and $\tau_A^* : (T_k^1)^*M \rightarrow T^*M$ is the projection on the A^{th} -copy T^*M of $(T_k^1)^*M$. Locally, $\omega_A = -d\theta_A = -d(p_i^A dx^i) = dx^i \wedge dp_i^A$.

If $Z \in \mathcal{X}(M)$ has the local 1-parametric group $h_s : Q \rightarrow Q$, then the *canonical lift* of Z to $(T_k^1)^*M$ is the vector field $Z^{C*} \in \mathcal{X}((T_k^1)^*M)$ whose local 1-parametric group is $(T_k^1)^*(h_s) : (T_k^1)^*M \rightarrow (T_k^1)^*M$. Locally, if $Z = Z^i \frac{\partial}{\partial x^i}$, then $Z^{C*} = Z^i \frac{\partial}{\partial x^i} - p_j^A \frac{\partial Z^j}{\partial x^k} \frac{\partial}{\partial p_k^A}$.

Definition 3.1. Let $T_k^1\mathcal{M} = T\mathcal{M} \oplus \dots \oplus T\mathcal{M}$ be the k -tangent bundle of a manifold \mathcal{M} .

1) A *k -vector field* on \mathcal{M} is a section $\mathbf{X} : \mathcal{M} \rightarrow T_k^1\mathcal{M}$ of $\tau : T_k^1\mathcal{M} \rightarrow \mathcal{M}$, the natural projection. A k -vector field \mathbf{X} on \mathcal{M} defines a family of vector fields X_1, \dots, X_k on \mathcal{M} by $X_A = \tau_A \circ \mathbf{X}$, where $\tau_A : T_k^1\mathcal{M} \rightarrow T\mathcal{M}$ is the projection on the A^{th} -copy $T\mathcal{M}$ of $T_k^1\mathcal{M}$.

- 2) An *integral section* of \mathbf{X} at a point $x \in \mathcal{M}$ is a map $\psi : U_0 \subset \mathbf{R}^k \rightarrow \mathcal{M}$, with $0 \in U_0$, such that $\phi(0) = x$, $\psi_*(t) \left(\frac{\partial}{\partial t^A}(t) \right) = X_A(\psi(t))$, for every $t \in U_0$.
3) A k -vector field \mathbf{X} is called *integrable* if there is an integral section at every point of \mathcal{M} .

Let $H : (T_k^1)^*M \rightarrow \mathbf{R}$ be a *Hamiltonian function*. The family $((T_k^1)^*M, \omega_A, H)$ is a *k-symplectic Hamiltonian system*. The *Hamilton-de Donder-Weyl (HDW) equations* are

$$\frac{\partial H}{\partial x^i}(\psi(t)) = - \sum_{A=1}^k \frac{\partial \psi_i^A}{\partial t^A}(t), \quad \frac{\partial H}{\partial p_i^A}(\psi(t)) = \frac{\partial \psi^i}{\partial t^A}(t), \quad (3.1)$$

where $\psi : \mathbf{R}^k \rightarrow (T_k^1)^*M$, $\psi(t) = (\psi^i(t), \psi_i^A(t))$, is a solution.

We denote by $\mathcal{X}_H^k((T_k^1)^*M)$ the set of k -vector fields on $(T_k^1)^*M$ solutions to

$$\sum_{A=1}^k i_{X_A} \omega_A = dH. \quad (3.2)$$

Any k -vector field (X_1, \dots, X_k) which is a solution of (3.2) will be called an *evolution k-vector field* associated with the Hamiltonian function H . It should be noticed that in general the solution to the above equation is not unique. Nevertheless, it can be proved [13] that there always exists an *evolution k-vector field* associated with a Hamiltonian function H .

Then, if $\mathbf{X} \in \mathcal{X}_H^k((T_k^1)^*M)$ is integrable and $\psi : \mathbf{R}^k \rightarrow (T_k^1)^*M$ is an integral section of \mathbf{X} , then $\psi(t) = (\psi^i(t), \psi_i^A(t))$ is a solution to the HDW equations (3.1).

In [5], [28], [27] it is introduced next definition for a conservation law of the Hamilton-de Donder-Weyl (HDW) equations (3.1) on $(T_k^1)^*M$:

Definition 3.2. ([28], [27]) A map $\Phi = (\Phi_1, \dots, \Phi_k) : (T_k^1)^*M \rightarrow \mathbf{R}^k$ is called **conservation law** for the Hamilton-de Donder-Weyl (HDW) equations (3.1) if the divergence of the function

$$\Phi \circ \psi = (\Phi_1 \circ \psi, \dots, \Phi_k \circ \psi) : U \subset \mathbf{R}^k \rightarrow \mathbf{R}^k$$

is zero, for every $\psi : U \subset \mathbf{R}^k \rightarrow M$ solution of the Hamilton-de Donder-Weyl (HDW) equations (3.1), that is

$$\sum_{A=1}^k \frac{\partial (\Phi_A \circ \psi)}{\partial t^A}(t) = 0. \quad (3.3)$$

In [27] was proved the next result:

Proposition 3.3. *If $\Phi = (\Phi_1, \dots, \Phi_k) : (T_k^1)^*M \rightarrow \mathbf{R}^k$ is a conservation law for the Hamilton-de Donder-Weyl (HDW) equations (3.1), then for every integrable k-vector field $\mathbf{X} = (X_1, \dots, X_k) \in \mathcal{X}_H^k((T_k^1)^*M)$ we have*

$$\sum_{A=1}^k L_{X_A} \Phi_A = 0. \quad (3.4)$$

The converse of proposition 3.3 may not be true and the reason is that we might have solutions ψ of the Hamilton-de Donder-Weyl (HDW) equations (3.1) that are not solutions to some $\mathbf{X} \in \mathcal{X}_H^k((T_k^1)^*M)$.

However, under some assumption, the converse is true ([5]):

Proposition 3.4. *If we assume that there exists a vector field $X \in \mathcal{X}((T_k^1)^*M)$ such that*

$$i_X(\omega_0)_A = df_A, \forall 1 \leq A \leq k \quad (3.5)$$

*for some functions $f_A : (T_k^1)^*M \rightarrow \mathbf{R}^k$, then $\mathbf{F} = (f_1, \dots, f_k)$ is a conservation law of HDW equations (3.1) if and only if $\sum_{A=1}^k X_A(f_A) = 0$, for every integrable k -vector field $\mathbf{X} = (X_1, \dots, X_k) \in \mathcal{X}_H^k((T_k^1)^*M)$.*

Definition 3.5. Let $((T_k^1)^*M, \omega_A, H)$ be a k -symplectic Hamiltonian system.

- 1) A *symmetry* is a diffeomorphism $\Phi : (T_k^1)^*M \rightarrow (T_k^1)^*M$ such that, for every solution ψ of the Hamilton-de Donder-Weyl (HDW) equations (3.1), we have that $\Phi \circ \psi$ is also a solution Hamilton-de Donder-Weyl (HDW) equations (3.1).
- 2) An *infinitesimal symmetry* is a vector field $Y \in \mathcal{X}((T_k^1)^*M)$ whose local flows are local symmetries.
- 3) A *Cartan symmetry* is a diffeomorphism $\Phi : (T_k^1)^*M \rightarrow (T_k^1)^*M$ such that $\Phi^*\omega_A = \omega_A$, for all $1 \leq A \leq k$, and $\Phi^*H = H$ (up to a constant).
- 4) An *infinitesimal Cartan symmetry* is a vector field $Y \in \mathcal{X}((T_k^1)^*M)$ such that $L_Y\omega_A$, for all $1 \leq A \leq k$, and $L_YH = 0$.

If $\Phi = (T_k^1)^*\phi$, for some diffeomorphism $\phi : M \rightarrow M$, then the (Cartan) symmetry Φ is said to be *natural*.

If $Y = Z^{C^*}$ for $Z \in \mathcal{M}$, then the infinitesimal (Cartan) symmetry is said to be *natural*.

If Φ is a Cartan symmetry then it is a symmetry.

If $\mathbf{X} = (X_1, \dots, X_k) \in \mathcal{X}_H^k((T_k^1)^*M)$, then $\Phi_*\mathbf{X} = (\Phi_*X_1, \dots, \Phi_*X_k) \in \mathcal{X}_H^k((T_k^1)^*M)$.

Proposition 3.6. ([27], [28]) *Let $Y \in \mathcal{X}((T_k^1)^*M)$ be an infinitesimal Cartan symmetry. Then, for every $p \in (T_k^1)^*M$, there is an open neighbourhood U_p , such that*

- i) *There exists $f_A \in C^\infty(U_p)$, unique up to a constant functions, such that $i_Y\omega_A = df_A$.*
- ii) *There exists $\zeta_A \in C^\infty(U_p)$, verifying $L_Y\theta_A = d\zeta_A$ on U_p and then $f_A = i_Y\theta_A - \zeta_A$ (up to a constant function on U_p).*

Theorem 3.7. (Noether Theorem) ([27], [28]) *Let $Y \in \mathcal{X}((T_k^1)^*M)$ be an infinitesimal Cartan symmetry.*

- i) *For every $p \in (T_k^1)^*M$, there is an open neighbourhood U_p , such that the functions $f_A = i_Y\theta_A - \zeta_A$, $1 \leq A \leq k$, define a conservation law $\mathbf{f} = (f_1, \dots, f_k)$ on U_p .*

- ii) *For every $\mathbf{X} = (X_1, \dots, X_k) \in \mathcal{X}_H^k((T_k^1)^*M)$, we have $\sum_{A=1}^k L_{X_A}f_A = 0$ on U_p .*

4 k -Symplectic Lagrangian formalism

Let be $T_k^1 M = TM \oplus \dots \oplus TM$ the k -tangent bundle of a manifold M , with natural projection $\tau : T_k^1 M \rightarrow M$. The natural coordinates on $T_k^1 M$ are (x^i, v_A^i) .

Locally, for $Z_x = a^i \frac{\partial}{\partial x^i} \in T_x M$ the *vertical A-lift* of Z at $(v_{1x}, \dots, v_{kx}) \in T_k^1 M$ is the vector field $(Z_x)^{VA} (v_{1x}, \dots, v_{kx}) = a^i \frac{\partial}{\partial v_A^i} |_{(v_{1x}, \dots, v_{kx})}$.

Locally, the *Liouville vector field* is $\Delta = \sum_{A=1}^k v_A^i \frac{\partial}{\partial v_A^i}$ and the k -tangent structure on $T_k^1 M$ is the set (S^1, \dots, S^k) of $(1, 1)$ -tensor fields defined by $S^A = \frac{\partial}{\partial v_A^i} \otimes dx^i$.

Let $Z \in \mathcal{X}(M)$ with the local 1-parametric group $h_s : Q \rightarrow Q$. Then the *canonical lift* of Z to $(T_k^1)_x M$ is the vector field $Z^C \in \mathcal{X}(T_k^1 M)$ whose local 1-parametric group is $T_k^1 h_s : T_k^1 M \rightarrow T_k^1 M$. Locally, if $Z = Z^i \frac{\partial}{\partial x^i}$, then $Z^C = Z^i \frac{\partial}{\partial x^i} + v_A^j \frac{\partial Z^k}{\partial x^j} \frac{\partial}{\partial v_A^k}$.

Definition 4.1. A second order partial differential equation (SOPDE) is a k -vector field Γ in $T_k^1 M$ which is a section of the projection $T_k^1 \tau : T_k^1(T_k^1 M) \rightarrow T_k^1 M$.

Locally, a SOPDE $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ is given by the vector fields $\Gamma_A = v_A^i \frac{\partial}{\partial x^i} + (\Gamma_A)^i_B \frac{\partial}{\partial v_B^i}$.

Proposition 4.2. If ψ is an integral section of an integrable SOPDE Γ , then $\psi = \phi^{(1)}$, where $\phi^{(1)}$ is the first prolongation of $\phi = \tau \circ \psi$ and ϕ is a solution to the system

$$\frac{\partial^2 \phi^i}{\partial t^A \partial t^B}(t) = (\Gamma_A)^i_B \left(\phi^j(t), \frac{\partial \phi^j}{\partial t^C}(t) \right). \quad (4.1)$$

Conversely, if $\phi : \mathbf{R}^k \rightarrow M$ is a solution of the system (4.1), then $\phi^{(1)}$ is an integral section of SOPDE Γ .

Let $L : T_k^1 M \rightarrow \mathbf{R}$ be a Lagrangian. The *Euler-Lagrange equations* for L are

$$\sum_{A=1}^k \frac{\partial}{\partial t^A}(t) \left(\frac{\partial L}{\partial v_A^i}(\phi(t)) \right) = \frac{\partial L}{\partial x^i}(\psi(t)), \quad v_A^i(\psi(t)) \frac{\partial \psi^i}{\partial t^A}(t), \quad (4.2)$$

whose solutions are maps $\psi : \mathbf{R}^k \rightarrow T_k^1 M$. We observe that $\psi(t) = \phi^{(1)}(t)$ for $\phi = \tau \circ \psi$.

We can introduce the forms associated to L , $(\theta_L)_A = dL \circ S^A \in \Omega^1(T_k^1 M)$, $(\omega_L)_A = -d(\theta_L)_A \in \Omega^2(T_k^1 M)$, and the *energy Lagrangian function* $E_L = \Delta(L) - L \in C^\infty(T_k^1 M)$. Locally,

$$(\theta_A)_L = \frac{\partial L}{\partial v_A^i} dx^i, \quad (\omega_A)_L = \frac{\partial^2 L}{\partial x^j \partial v_A^i} dx^i \wedge dx^j + \frac{\partial^2 L}{\partial v_B^j \partial v_A^i} dx^i \wedge dv_B^j, \quad E_L = v_A^i \frac{\partial L}{\partial v_A^i} - L.$$

The Lagrangian L is *regular* if the matrix $\left(\frac{\partial^2 L}{\partial v_A^i \partial v_B^j}\right)$ is regular at every point of $T_k^1 M$.

This is equivalent to say that $((\omega_L)_1, \dots, (\omega_L)_k; V)$, $V = \ker \tau_*$, is a k -symplectic structure.

The family $(T_k^1 M, (\omega_L)_A, E_L)$ is called a *k -symplectic Lagrangian system*.

Let $\mathcal{X}_L^k(T_k^1 M)$ be the set of k -vector fields $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ in $T_k^1 M$ solutions to

$$\sum_{A=1}^k i_{\Gamma_A} (\omega_L)_A = dE_L. \quad (4.3)$$

Locally, if $\Gamma_A = (\Gamma_A)^i \frac{\partial}{\partial x^i} + (\Gamma_A)^j_B \frac{\partial}{\partial v_B^j}$ and L is regular, then Γ is a solution of (4.3) if and only if

$$\frac{\partial^2 L}{\partial x^j \partial v_A^i} v_A^j + \frac{\partial^2 L}{\partial v_A^i \partial v_B^j} (\Gamma_A)^j_B = \frac{\partial L}{\partial x^i}, \quad (\Gamma_A)^i = v_A^i.$$

Thus, if $\Gamma \in \mathcal{X}_L^k(T_k^1 M)$, then it is a SOPDE and, if it is integrable, its integral sections are first prolongations of maps $\phi : \mathbf{R}^k \rightarrow M$ solution to the Euler-Lagrange equations (4.2).

Now, if we consider a regular Lagrangian on $T_k^1 M$, then we can rewrite the results from above for the Hamiltonian system $(T_k^1 M, (\omega_L)_A, E_L)$ with Hamiltonian function $H = E_L = v_A^i \frac{\partial L}{\partial v_A^i} - L$.

In the classical case ($k = 1$), let us recall that Cartan symmetries induce and are induced by constants of motions (conservation laws), and these results are known as Noether Theorem and its converse ([6], [7], [11], [23], [24]).

For the higher order case the problem was solved by L. Bua, I. Bucătaru and M. Salgado in [5]. So, for $k > 1$ the Noether Theorem is also true, that is each Cartan symmetry induces a conservation law (defined for a regular Lagrangian on $T_k^1 M$, like in [5]). However, the converse of Noether Theorem may not be true and in [5] is provided some examples of conservation laws that are not induced by Cartan symmetries.

In [5], [27] it is introduced next definition for a conservation law of Euler-Lagrange equations (4.2) on $T_k^1 M$:

Definition 4.3. ([5]) A map $\Phi = (\Phi_1, \dots, \Phi_k) : T_k^1 M \rightarrow \mathbf{R}^k$ is called **conservation law** for the Euler-Lagrange equations (4.2) if the divergence of

$$\Phi \circ \phi^{(1)} = \left(\Phi_1 \circ \phi^{(1)}, \dots, \Phi_k \circ \phi^{(1)} \right) : U \subset \mathbf{R}^k \rightarrow \mathbf{R}^k$$

is zero, for every $\phi : U \subset \mathbf{R}^k \rightarrow M$ solutions of the Euler-Lagrange equations (4.2), that is

$$\sum_{A=1}^k \frac{\partial (\Phi_A \circ \phi^{(1)})}{\partial t^A}(t) = 0. \quad (4.4)$$

In [5] was proved the next results:

Proposition 4.4. *Let $\Phi = (\Phi_1, \dots, \Phi_k) : T_k^1 M \longrightarrow \mathbf{R}^k$ be a conservation law for the Euler-Lagrange equations (4.2). If $\xi = (\xi_1, \dots, \xi_k)$ is an integrable SOPDE which belongs to $\mathcal{X}_L^k(T_k^1 M)$, then*

$$\sum_{A=1}^k L_{\xi_A} \Phi_A = 0. \quad (4.5)$$

The converse of proposition 4.4 may not be true and the reason is that we might have solutions ϕ of the Euler-Lagrange equations (4.2) that are not solutions to some $\xi \in \mathcal{X}_L^k(T_k^1 M)$.

However, under some assumption, the converse is true ([5]):

Proposition 4.5. *Let L be a regular Lagrangian on $T_k^1 M$ and assume that there exists a vector field $X \in \mathcal{X}(T_k^1 M)$ such that*

$$i_X(\omega_L)_A = df_A, \forall 1 \leq A \leq k \quad (4.6)$$

for some functions $f_A : T_k^1 M \longrightarrow \mathbf{R}^k$.

Then $\mathbf{F} = (f_1, \dots, f_k)$ is a conservation law for the Euler-Lagrange equations (4.2) if and only if $\sum_{A=1}^k \xi_A(f_A) = 0$, for all integrable SOPDE $\xi \in \mathcal{X}_L^k(T_k^1 M)$.

Using the notions from the previous section, we have:

Definition 4.6. ([5]) A vector field $X \in \mathcal{X}(T_k^1 M)$ is called a Cartan symmetry for the regular Lagrangian L , if $L_X(\omega_L)_A = 0$, for all $1 \leq A \leq k$ and $L_X E_L = 0$.

In this case the flow ϕ_t of X transforms solutions of the Euler-Lagrange equations on solutions of the Euler-Lagrange equations, that is, each ϕ_t is a symmetry of the Euler-Lagrange equations ([5], [27]).

Let us remark that for a Cartan symmetry of L , $L_X(\omega_L)_A = 0$ implies that, locally, we have $i_X(\omega_L)_A = df_A$, for all $1 \leq A \leq k$ ([5]). So, if X is a Cartan symmetry, then proposition 4.5 holds locally.

Theorem 4.7. (Noether Theorem) ([5]) *Let L be a regular Lagrangian on $T_k^1 M$ and $X \in \mathcal{X}(T_k^1 M)$ a Cartan symmetry for L . Then, there exists (locally defined) functions f_A on $T_k^1 M$ such that*

$$L_X(\theta_L)_A = df_A, \quad 1 \leq A \leq k, \quad (4.7)$$

and the following functions

$$\Phi_A = (\theta_L)_A(X) - f_A, \quad 1 \leq A \leq k, \quad (4.8)$$

give a conservation law for the Euler-Lagrange equations associated to L , i.e. for any integrable evolution k -vector field (defined locally) associated to $H = E_L$, the energy of L .

Next result show when a conservation law for a Lagrangian induces and are induced by a Cartan symmetry.

Theorem 4.8. ([5]) Let be L a regular Lagrangian on $T_k^1 M$, the functions $f_A \in C^\infty(T_k^1 M)$, $1 \leq A \leq k$, and a vector field $X \in \mathcal{X}(T_k^1 M)$ such that

$$i_X(\omega_L)_A = df_A, 1 \leq A \leq k. \quad (4.9)$$

Then $F = (f_1, \dots, f_k)$ is a conservation law for L if and only if X is a Cartan symmetry.

Example 4.9. ([5], [19]) The equation of motion of a vibrating string is

$$\sigma \frac{\partial^2 \phi}{\partial (t^1)^2} - \tau \frac{\partial^2 \phi}{\partial (t^2)^2} = 0, \quad (4.10)$$

where σ and τ are certain constants of the mechanical system.

The equation (4.10) can be described as the generalized Euler-Lagrange equations associated to a Lagrangian $L(x, v_1, v_2) = \frac{1}{2}(\sigma v_1^2 - \tau v_2^2)$ defined on the jet bundle $T_k^1 M$ with $M = \mathbf{R}$ and $k = 2$. Since L is regular there exists a k -symplectic structure $((\omega_L)_1, (\omega_L)_2)$, associated to L , given in local coordinates by $(\omega_L)_1 = \sigma dv_1 \wedge dx$, $(\omega_L)_2 = -\tau dv_2 \wedge dx$. The energy $E_L = \Delta(L) - L$ is locally given by $E_L = \frac{1}{2}(\sigma v_1^2 - \tau v_2^2)$ and $dE_L = \sigma v_1 dv_1 - \tau v_2 dv_2$.

Let us suppose that there exists $\xi = (\xi_1, \xi_2) \in T_2^1(T_2^1 \mathbf{R})$ a solution of the equation

$$i_{X_1}(\omega_L)_1 + i_{X_2}(\omega_L)_2 = dE_L. \quad (4.11)$$

Then, (ξ_1, ξ_2) is a SOPDE and, locally, $\xi_A = v_A \frac{\partial}{\partial x} + (\xi_A)_1 \frac{\partial}{\partial v_1} + (\xi_A)_2 \frac{\partial}{\partial v_2}$, $A = 1, 2$. So, we have $\sigma(\xi_1)_1 - \tau(\xi_2)_2 = 0$ and if we consider $\phi : \mathbf{R}^2 \rightarrow \mathbf{R}$, $\phi = \phi(t^1, t^2)$, a solution of $\xi = (\xi_1, \xi_2)$, then we obtain $0 = \sigma(\xi_1)_1 - \tau(\xi_2)_2 = \sigma \frac{\partial^2 \phi}{\partial (t^1)^2} - \tau \frac{\partial^2 \phi}{\partial (t^2)^2}$. Thus, the equation (4.11) is a geometric version for the equations (4.10). An example of an integrable SOPDE solution $\xi = (\xi_1, \xi_2)$ of (4.10) is given by ([5])

$$\begin{aligned} \xi_1 &= v_1 \frac{\partial}{\partial x} + \tau(\sigma(v_1)^2 + \tau(v_2)^2) \frac{\partial}{\partial v_1} + 2\sigma\tau v_1 v_2 \frac{\partial}{\partial v_2}, \\ \xi_2 &= v_2 \frac{\partial}{\partial x} + 2\sigma\tau v_1 v_2 \frac{\partial}{\partial v_1} + \sigma(\sigma(v_1)^2 + \tau(v_2)^2) \frac{\partial}{\partial v_2}. \end{aligned}$$

Thus any solution ϕ of the SOPDE $\xi = (\xi_1, \xi_2)$ in the formulae above is a solution of the vibrating string equation (4.10). The following two functions $\Phi_1, \Phi_2 : T_2^1 \mathbf{R} \rightarrow \mathbf{R}$,

$$\Phi_1(v_1, v_2) = -2\sigma v_1 v_2, \quad \Phi_2(v_1, v_2) = \sigma(v_1)^2 + \tau(v_2)^2 \quad (4.12)$$

give a conservation law $\Phi = (\Phi_1, \Phi_2)$ for every evolution 2-vector field associated with the Hamiltonian E_L . We can say that $\Phi = (\Phi_1, \Phi_2)$ is a conservation law for the Euler-Lagrange equations (4.10) of the vibrating string, or $\Phi = (\Phi_1, \Phi_2)$ give a conservation law for an integrable evolution k -vector field associated to $H = E_L$. More that, this conservation law is not induced by a Cartan symmetry, and hence it will show that the converse of the Noether Theorem 4.7 is not true, unless the assumptions (4.9) are satisfied ([5]).

5 New kinds of conservation laws for k -symplectic systems

In this section we will present a result which allow us to obtain new kinds of conservation laws (nonclassical) for k -symplectic Hamiltonian and Lagrangian systems, without the help of a Noether type theorem and without the use of a variational principle, using only symmetries and pseudosymmetries associated to the k -vector fields $\mathbf{X} = (X_1, \dots, X_k)$ which are solutions of the equation $\sum_{A=1}^k i_{X_A} \omega_A = dH$ ([21]). This result is a generalization from the classical case of a results of G.L. Jones ([11]) and M. Crăsmăreanu ([7]).

Let \mathcal{M} be a manifold and $\mathbf{X} = (X_1, \dots, X_k)$ be a k -vector field on \mathcal{M} .

Definition 5.1. The map $\Phi = (\Phi_1, \dots, \Phi_k) : \mathcal{M} \longrightarrow \mathbf{R}^k$ is called *conservation law* for $\mathbf{X} = (X_1, \dots, X_k)$ if

$$\sum_{A=1}^k L_{X_A} \Phi_A = 0. \quad (5.1)$$

Definition 5.2. A vector field Y on \mathcal{M} is called *symmetry* or *dynamical symmetry* of the k -vector field $\mathbf{X} = (X_1, \dots, X_k)$ if

$$L_{X_A} Y = 0, \forall \quad A = 1, \dots, k. \quad (5.2)$$

Definition 5.3. A vector field Y on \mathcal{M} is called *pseudosymmetry* or *dynamical pseudosymmetry* of the k -vector field $\mathbf{X} = (X_1, \dots, X_k)$ if, for all $A = 1, \dots, k$, there are functions $\lambda_A^B \in C^\infty(\mathcal{M})$, $B = 1, \dots, k$, such that

$$L_{X_A} Y = \sum_{B=1}^k \lambda_A^B X_B. \quad (5.3)$$

This definitions is according to Krupková definition for symmetry of a distribution ([12]). Furthermore, we can give a generalization of this notion:

Definition 5.4. If we fixed a k -vector field $\mathbf{Z} = (Z_1, \dots, Z_k)$ on \mathcal{M} , then a vector field Y on \mathcal{M} is called *\mathbf{Z} -pseudosymmetry* of the k -vector field $\mathbf{X} = (X_1, \dots, X_k)$ if, for all $A = 1, \dots, k$, there are functions $\lambda_A^B \in C^\infty(\mathcal{M})$, $B = 1, \dots, k$, such that

$$L_{X_A} Y = \sum_{B=1}^k \lambda_A^B Z_B. \quad (5.4)$$

Obviously, a X -pseudosymmetry of $\mathbf{X} = (X_1, \dots, X_k)$ is a pseudosymmetry of \mathbf{X} and a \mathbf{O} -pseudosymmetry of \mathbf{X} is a symmetry for \mathbf{X} .

Next, we will present the main result of the paper, which allow us to obtain new kinds of conservation laws for k -symplectic Hamiltonian and Lagrangian systems, without the help of a Noether type theorem and without the use of a variational principle.

Theorem 5.5. Let $\mathbf{X} = (X_1, \dots, X_k)$ a k -vector field on \mathcal{M} and $(\omega_1, \dots, \omega_k)$ be a family of p -forms on \mathcal{M} , invariant for \mathbf{X} , i.e. $L_{X_A} \omega_A = 0$, for all $A = 1, \dots, k$. Let Y be a symmetry of \mathbf{X} and the k -vector field $\mathbf{Y} = (Y, \dots, Y)$. If we have $p-1$ vector fields on \mathcal{M} , S_1, \dots, S_{p-1} , which are \mathbf{Y} -pseudosymmetries of \mathbf{X} , then

$$\Phi = (\Phi_1, \dots, \Phi_k), \quad (5.5)$$

is a conservation law for $\mathbf{X} = (X_1, \dots, X_k)$, where $\Phi_A = \omega_A(S_1, \dots, S_{p-1}, Y)$, $A = 1, \dots, k$, or locally,

$$\Phi_A = (S_1)^{i_1} \dots (S_{p-1})^{i_{p-1}} (Y)^{i_p} (\omega_A)_{i_1 \dots i_{p-1} i_p}. \quad (5.6)$$

Particularly, if Y, S_1, \dots, S_{p-1} are symmetries for \mathbf{X} then Φ given by (5.5) is a conservation law for \mathbf{X} .

Proof. Applying the properties of the Lie derivative and taking into account that, locally, $\Phi_A = (S_1)^{i_1} \dots (S_{p-1})^{i_{p-1}} (Y)^{i_p} (\omega_A)_{i_1 \dots i_{p-1} i_p}$, for all $A = 1, \dots, k$, we obtain that

$$\begin{aligned} \sum_{A=1}^k L_{X_A} \Phi_A &= \sum_{A=1}^k L_{X_A} \left((S_1)^{i_1} \dots (S_{p-1})^{i_{p-1}} (Y)^{i_p} (\omega_A)_{i_1 \dots i_{p-1} i_p} \right) = \\ &= \sum_{A=1}^k (L_{X_A} S_1)^{i_1} (S_2)^{i_2} \dots (S_{p-1})^{i_{p-1}} (Y)^{i_p} (\omega_A)_{i_1 \dots i_{p-1} i_p} + \dots + \\ &+ \sum_{A=1}^k (S_1)^{i_1} \dots (S_{p-2})^{i_{p-2}} (L_{X_A} S_{p-1})^{i_{p-1}} (Y)^{i_p} (\omega_A)_{i_1 \dots i_{p-1} i_p} + \\ &+ \sum_{A=1}^k (S_1)^{i_1} \dots (S_{p-1})^{i_{p-1}} (L_{X_A} Y)^{i_p} (\omega_A)_{i_1 \dots i_{p-1} i_p} + \\ &+ \sum_{A=1}^k (S_1)^{i_1} \dots (S_{p-1})^{i_{p-1}} (Y)^{i_p} (L_{X_A} \omega_A)_{i_1 \dots i_{p-1} i_p} = \\ &= \sum_{A=1}^k \left(\sum_{B=1}^k (\lambda_1)_A^B \right) (Y)^{i_1} (S_2)^{i_2} \dots (S_{p-1})^{i_{p-1}} (Y)^{i_p} (\omega_A)_{i_1 \dots i_{p-1} i_p} + \dots \\ &+ \sum_{A=1}^k (S_1)^{i_1} \dots (S_{p-2})^{i_{p-2}} \left(\sum_{B=1}^k (\lambda_{p-1})_A^B \right) (Y)^{i_{p-1}} (Y)^{i_p} (\omega_A)_{i_1 \dots i_{p-1} i_p} = 0, \end{aligned}$$

because $L_{X_A} Y = 0$, $L_{X_A} \omega_A = 0$ and taking into account of the antisymmetry of ω_A . \square

As an immediate consequence of the previous theorem, we have the result:

Theorem 5.6. Let $(\mathcal{M}, \omega_A, V; 1 \leq A \leq k)$ be a k -symplectic manifold and $H : \mathcal{M} \rightarrow \mathbb{R}$ be a function on \mathcal{M} . Let $\mathbf{X} = (X_1, \dots, X_k)$ be an integrable evolution k -vector field associated to H , i.e. $\mathbf{X} \in \mathcal{X}_H^k(\mathcal{M})$. Let us suppose that $L_{X_A} \omega_A = 0$, for all $A = 1, \dots, k$. Let Y be a symmetry of \mathbf{X} and the k -vector field $\mathbf{Y} = (Y, \dots, Y)$. If we have a vector field S on \mathcal{M} which is a \mathbf{Y} -pseudosymmetry of \mathbf{X} , then

$$\Phi = (\Phi_1, \dots, \Phi_k),$$

is a conservation law for $\mathbf{X} = (X_1, \dots, X_k)$, where $\Phi_A = \omega_A(S, Y)$, for all $A = 1, \dots, k$.

Particularly, if Y, S are symmetries for \mathbf{X} then Φ is a conservation law for $\mathbf{X} = (X_1, \dots, X_k)$.

Remark 5.7. a) Obviously, for any k -vector field $\mathbf{X} \in \mathcal{X}_H^k(\mathcal{M})$, using (3.2), we have $\sum_{A=1}^k L_{X_A} \omega_A = 0$. But, for our purpose we need $L_{X_A} \omega_A = 0$, $A = 1, \dots, k$.

b) The Hamiltonian function H is not a conservation law for an integrable evolution k -vector field $\mathbf{X} = (X_1, \dots, X_k) \in \mathcal{X}_H^k(\mathcal{M})$. Neither the map $\mathbf{H} = (H, \dots, H) : \mathcal{M} \rightarrow \mathbf{R}^k$ is not a conservation law for any integrable evolution k -vector field $\mathbf{X} \in \mathcal{X}_H^k(\mathcal{M})$.

Now, using this last result we can obtain new kinds of conservation laws for k -symplectic Hamiltonian systems and k -symplectic Lagrangian systems.

Corollary 5.8. Let $((T_k^1)^*M, (\omega_0)_A, H)$ be a k -symplectic Hamiltonian system and $\mathbf{X} = (X_1, \dots, X_k)$ be an integrable evolution k -vector field associated to H , i.e. $\mathbf{X} \in \mathcal{X}_H^k((T_k^1)^*M)$. Let us suppose that $L_{X_A}(\omega_0)_A = 0$, for all $A = 1, \dots, k$. Let $Y \in \mathcal{X}((T_k^1)^*M)$ be a symmetry of \mathbf{X} and the k -vector field $\mathbf{Y} = (Y, \dots, Y)$ on $(T_k^1)^*M$. If we have a vector field S on $(T_k^1)^*M$ which is a \mathbf{Y} -pseudosymmetry of \mathbf{X} , then

$$\Phi = (\Phi_1, \dots, \Phi_k),$$

is a conservation law for $\mathbf{X} = (X_1, \dots, X_k)$, where $\Phi_A = (\omega_0)_A(S, Y)$, for all $A = 1, \dots, k$.

Particularly, if \mathbf{Y}, S are symmetries for \mathbf{X} then Φ is a conservation law for $\mathbf{X} = (X_1, \dots, X_k)$.

Corollary 5.9. Let $(T_k^1M, (\omega_L)_A, E_L)$ be a k -symplectic Lagrangian system and $\mathbf{X} = (X_1, \dots, X_k)$ be an integrable evolution k -vector field associated to $H = E_L$, i.e. $\mathbf{X} \in \mathcal{X}_L^k(T_k^1M)$. Let us suppose that $L_{X_A}(\omega_L)_A = 0$, for all $A = 1, \dots, k$. Let $Y \in \mathcal{X}(T_k^1M)$ be a symmetry of \mathbf{X} and the k -vector field $\mathbf{Y} = (Y, \dots, Y)$ on T_k^1M . If we have a vector field S on T_k^1M which is a \mathbf{Y} -pseudosymmetry of \mathbf{X} , then

$$\Phi = (\Phi_1, \dots, \Phi_k),$$

is a conservation law for $\mathbf{X} = (X_1, \dots, X_k)$, where $\Phi_A = (\omega_L)_A(S, Y)$, for all $A = 1, \dots, k$.

Particularly, if \mathbf{Y}, S are symmetries for \mathbf{X} then Φ is a conservation law for $\mathbf{X} = (X_1, \dots, X_k)$.

Remark 5.10. If each vector fields X_1, \dots, X_k of $\mathbf{X} \in \mathcal{X}_L^k(T_k^1M)$ are Cartan symmetries for L , then we have $L_{X_A}(\omega_L)_A = 0$, for all $A = 1, \dots, k$, and then we can apply the last corollary for this k -vector field \mathbf{X} . Moreover, we have that (H, \dots, H) is a conservation law for $\mathbf{X} = (X_1, \dots, X_k)$, where $H = E_L$.

Example 5.11. ([5], [24]) a) If we consider the Lagrangians $L_1, L_2 : T_2^1 \mathbf{R} \rightarrow \mathbf{R}$,

$$L_1(x, v_1, v_2) = \frac{1}{2}(\sigma(v_1)^2 - \tau(v_2)^2), \quad L_2(x, v_1, v_2) = \sqrt{1 + (v_1)^2 + (v_2)^2},$$

then the vector field $X = \frac{\partial}{\partial x}$ is a Cartan symmetry for L_1 and L_2 . The induced conservation laws are $\Phi = (\Phi_1 = \sigma v_1, \Phi_2 = -\tau v_2)$ for L_1 and $\Phi = (\Phi_1 = \frac{v_1}{\sqrt{1+(v_1)^2+(v_2)^2}}, \Phi_2 = \frac{v_2}{\sqrt{1+(v_1)^2+(v_2)^2}})$ for L_2 .

Let us observe that the above Lagrangians correspond to the vibrating string equations and, respectively to the equations of minimal surfaces.

b) For the Lagrangian $L : T_3^1 \mathbf{R} \rightarrow \mathbf{R}$, defined by

$$L(x, v_1, v_2, v_3) = \frac{1}{2}((v_1)^2 + (v_2)^2 + (v_3)^2),$$

the vector field $X = \frac{\partial}{\partial x}$ is a Cartan symmetry, and the induced conservation law is $\Phi = (\Phi_1, \Phi_2, \Phi_3)$, where $\Phi_i = v_i$, $i = 1, 2, 3$. The Euler-Lagrange equations corresponding to L are the Laplace's equations.

c) For the Lagrangian $L : T_2^1 \mathbf{R}^2 \rightarrow \mathbf{R}$, defined by

$$L(x^1, x^2, v_1^1, v_2^1, v_1^2, v_2^2) = \left(\frac{1}{2}\lambda + \nu\right) [(v_1^1)^2 + (v_2^2)^2] + \frac{1}{2}\nu [(v_2^1)^2 + (v_1^2)^2] + (\lambda + \nu)v_1^1 v_2^2,$$

the vector field $X = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}$ is a Cartan symmetry, and the induced conservation law is $\Phi = (\Phi_1, \Phi_2)$, where $\Phi_1 = (\lambda + 2\nu)v_1^1 + \nu v_1^2 + (\lambda + \nu)v_2^2$, $\Phi_2 = (\lambda + \nu)v_1^1 + \nu v_2^1 + (\lambda + 2\nu)v_2^2$. The Euler-Lagrange equations corresponding to L are the Navier's equations.

Acknowledgments. This research was supported by FP7-PEOPLE-2012-IRSES-316338.

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